Lecture 26

Plan: § 9.5 Homogeneous Linear system

In this lecture, we will discuss how to solve constant coefficients homogeneous linear system:

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

where $A = [a_{ij}]_{1\le i,j\le n}$, $a_{ij} \in \mathbb{R}$.

Recall last lecture: If $\vec{x} = e^{\lambda t}\vec{v}$ is a soln to "$\frac{d\vec{x}}{dt} = A\vec{x}$" $\Longleftrightarrow$

$$A\vec{v} = \lambda\vec{v} \qquad (*)$$

$\underline{\text{Def}^n}$: If $\lambda \in \mathbb{R}$, $\vec{v} \neq 0$ satisfy "$A\vec{v} = \lambda \vec{v}$"

then λ is called an eigenvalue of A

$\vec{v}$ is called an eigenvector of A associated to λ.

Recall from linear algebra:

- "$\exists \vec{v} \neq 0$ such that $A\vec{v} = \lambda\vec{v}$."

$\Longleftrightarrow$ λ is an eigenvalue of A

$\Longleftrightarrow$ $\det(\lambda \underset{\substack{\downarrow \\ \text{identity matrix}}}{I_n} - A) = 0$

- $\det(\lambda I_n - A)$ is a polynomial of degree n in λ

— Called the characteristic polynomial of A

- To find all eigenvalues of A $\Longleftrightarrow$ to find all roots of the characteristic polynomial: $\det(\lambda I_n - A) = 0$

The above motivates the following algorithm to find the general solutions.

Algorithm to solve "$\frac{d\vec{x}}{dt} = A\vec{x}$":

Step 1: Compute $\det(\lambda I_n - A)$, which is a polynomial of λ of degree n. Find the roots of this polynomial: $\underbrace{\lambda_1, \cdots, \lambda_n}_{\text{Not necessarily distinct}}$.

They are the eigenvalues of A

Step 2: Find n "(linearly independent)" eigenvectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ of A associated to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively.

Step 3: Then the following gives a fundamental solution set:

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$\vec{x}_1 = e^{\lambda_1 t}\vec{v_1}, \quad \cdots, \quad \vec{x}_n = e^{\lambda_n t}\vec{v_n}.$

The general soln to "$\frac{d\vec{x}}{dt} = A\vec{x}$" is

$$\vec{x} = c_1 \vec{x}_1 + \cdots + c_n \vec{x}_n$$

$$= c_1 e^{\lambda_1 t} \vec{v}_1 + \cdots + c_n e^{\lambda_n t} \vec{v}_n,$$

$c_1, \cdots, c_n \in \mathbb{R}$

E.g.: Find the general soln to

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}_{2\times 2} \vec{x}$$

A: write $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Step 1: By solving $\det(\lambda I_2 - A) = 0$, find the eigenvalues of A. $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

Note $\lambda I_2 - A = \begin{bmatrix} \lambda-2 & +3 \\ -1 & \lambda+2 \end{bmatrix}_{2\times 2}$

$$\Rightarrow \quad \det(\lambda I_2 - A) = (\lambda-2)(\lambda+2) - (-3)$$

$$= \lambda^2 - 1$$

Let $\det(\lambda I_2 - A) = 0 \Rightarrow$

$$\lambda_1 = -1, \quad \lambda_2 = 1$$

Hence the eigenvalues of A are

$$\lambda_1 = -1, \qquad \lambda_2 = 1$$

Step 2: compute respectively the eigenvectors $\vec{v}_1, \vec{v}_2$ of A associated to λ_1, λ_2

① Find an eigenvector $\vec{v}_1$ associated to $\lambda_1 = -1$.

We need to find $\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 = (-1)\vec{v}_1$$

$$\Leftrightarrow \quad A\vec{v}_1 = -I_2 \vec{v}_1$$

$$\Leftrightarrow \quad (-I_2 - A)\vec{v}_1 = 0$$

(In general, $(\lambda I_n - A)\vec{v} = 0$)

$$\Leftrightarrow \quad \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \begin{cases} -3x + 3y = 0 \\ -x + y = 0 \end{cases} \Leftrightarrow \quad -x + y = 0$$

$$\Leftrightarrow \quad y = x \qquad \text{pick } x = 1, \Rightarrow y = 1$$

or any nonzero number as you like

e.g.: $x=2 \Rightarrow y=2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\Rightarrow \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

② Find an eigenvector $\vec{v}_2$ associated to $\lambda_2 = 1$

That is, find $\vec{V_2} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$A\vec{V_2} = \lambda_2 \vec{V_2} = \vec{V_2}$$

$$\Rightarrow \quad A\vec{V_2} = I_2 \vec{V_2}$$

$$\Rightarrow \quad (I_2 - A)\vec{V_2} = 0$$

$$\Rightarrow \quad \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \begin{cases} -x+3y = 0 \\ -x+3y = 0 \end{cases} \quad \Leftrightarrow -x+3y=0$$

$$\Leftrightarrow \quad x = 3y \qquad \text{pick } y=1 \Rightarrow x=3$$

or any nonzero number

$$\Rightarrow \quad \vec{V_2} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Note: linear algebra:

If eigenvalues of A $\lambda_1 \neq \lambda_2$

$\Rightarrow$ their eigenvectors $\vec{V_1}, \vec{V_2}$ are linearly independent.

$\Rightarrow e^{\lambda_1 t}\vec{V_1}, e^{\lambda_2 t}\vec{V_2}$ are linearly independent

Summarize:

$$\lambda_1 = -1 \rightarrow \vec{V_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \rightarrow \vec{V_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Step 3. Write down the general soln:

$$X = c_1 e^{\lambda_1 t}\vec{V_1} + c_2 e^{\lambda_2 t}\vec{V_2}$$

$$= c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{-t} + 3c_2 e^{t} \\ c_1 e^{-t} + c_2 e^{t} \end{bmatrix}$$

E.g Solve the following I.V.P for the linear system:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \vec{x} \qquad (*)$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

A: First we find the general soln to the linear system $(*)$.

We did this in the above E.g:

$$\vec{x} = \begin{bmatrix} c_1 e^{-t} + 3c_2 e^{t} \\ c_1 e^{-t} + c_2 e^{t} \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

plug in the initial condition $\vec{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

when $t=0$, $\vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ c_1 + c_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 + 3c_2 = 1 \\ c_1 + c_2 = 5 \end{cases}$$

$$\Rightarrow c_1 = 7, \quad c_2 = -2$$

Hence the soln to the I.V.P:

$$\vec{x} = \begin{bmatrix} 7e^{-t} - 6e^{t} \\ 7e^{-t} - 2e^{t} \end{bmatrix}$$

E.g: Solve the I.V.P of the linear system:

$$\begin{cases} \vec{x}' = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}_{3\times 3} \vec{x} \\ \vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

A: write $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

Step 1: find the eigenvalues of A by solving $\det(\lambda I_3 - A) = 0$.

Note

$$\lambda I_3 - A = \begin{bmatrix} \lambda-1 & -2 & 1 \\ -1 & \lambda & -1 \\ -4 & 4 & \lambda-5 \end{bmatrix}$$

$\Rightarrow$ $\det(\lambda I_3 - A)$

$$= (\lambda-1)\begin{vmatrix} \lambda & -1 \\ 4 & \lambda-5 \end{vmatrix} + 2\begin{vmatrix} -1 & -1 \\ -4 & \lambda-5 \end{vmatrix} + \begin{vmatrix} -1 & \lambda \\ -4 & 4 \end{vmatrix}$$

$$= (\lambda-1)(\lambda^2-5\lambda+4) + 2(-\lambda+5-4) + (-4+4\lambda)$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

(Hint: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda-2)(\lambda-3)$)

Hence $\det(\lambda I_3 - A) = 0 \Rightarrow$

$$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3.$$

Step 2: Find eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of A associated to $\lambda_1, \lambda_2, \lambda_3$

(1) To find $\vec{v}_1$ assoc. to $\lambda_1 = 1$, we write

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and}$$

solve $(\lambda_1 I_3 - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -2y + z = 0 \\ -x + y - z = 0 \\ -4x + 4y - 4z = 0 \end{cases} \Rightarrow \begin{cases} -2y + z = 0 \\ -x + y - z = 0 \end{cases}$$

$\Rightarrow z = 2y, \quad x = y - z = -y$

pick $y = 1$, $\Rightarrow x = -1$, $z = 2$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

(2) To find $\vec{v}_2$ asso. to $\lambda_2 = 2$

write $\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Solve $(\lambda_2 I_3 - A)\vec{v}_2 = 0$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} x - 2y + z = 0 \\ -x + 2y - z = 0 \\ -4x + 4y - 3z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x-2y+z=0 & ① \\ -4x+4y-3z=0 & ② \end{cases}$$

$2①+②$ $\Rightarrow -2x-z=0 \Rightarrow z=-2x$

By ①, $y=\frac{1}{2}(x+z)=\frac{1}{2}(-x)=-\frac{1}{2}x$

Let $x=1$. $\Rightarrow \vec{V_2}=\begin{bmatrix} 1 \\ -\frac{1}{2} \\ -2 \end{bmatrix}$

(3) To find $\vec{V_3}$ asso. to λ_3.

Write $\vec{V_3}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Solve $(\lambda_3 I_3 - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ -4 & 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$$\Rightarrow \begin{cases} -2x-2y+z=0 \\ -x-3y-z=0 \\ -4x+4y-2z=0 \end{cases} \Rightarrow \begin{cases} 2x-2y+z=0 & ① \\ -x+3y-z=0 & ② \end{cases}$$

$\Rightarrow x+y=0 \Rightarrow y=-x$

By ①, $\Rightarrow z=-2x+2y=-4x$

Let $x=1 \Rightarrow \vec{V_3}=\begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$

Summarize:

$$\lambda_1 = 1 \longrightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 2 \longrightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -2 \end{bmatrix}$$

$$\lambda_3 = 3 \rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

Step 3: Write down the general soln.

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3$$

$$= c_1 e^{t} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} -c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t} \\ c_1 e^{t} + -\frac{1}{2} c_2 e^{2t} - c_3 e^{3t} \\ 2c_1 e^{t} - 2c_2 e^{2t} - 4c_3 e^{3t} \end{bmatrix}$$

Step 4: plug in the initial condition $\vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

plug in $t=0$, $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -c_1 + c_2 + c_3 \\ c_1 - \frac{1}{2}c_2 - c_3 \\ 2c_1 - 2c_2 - 4c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -c_1 + c_2 + c_3 = -1 & (1) \\ c_1 - \frac{1}{2}c_2 - c_3 = 0 & (2) \\ 2c_1 - 2c_2 - 4c_3 = 0 & (3) \end{cases}$$

$(1) + (2) \Rightarrow c_2 = -2$

$2(1) + (3) \Rightarrow c_3 = 1$

$\Rightarrow c_1 = 0$

$\Rightarrow$ The soln to the I.V.P:

$$\vec{x} = \begin{bmatrix} -2e^{2t} + e^{3t} \\ e^{2t} - e^{3t} \\ 4e^{2t} - 4e^{3t} \end{bmatrix}$$

E.g: Find a general solution of

$\vec{x}' = A\vec{x}$, where $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

A: Step 1: Find eigenvalues of A

Note $\det(\lambda I_3 - A) = \begin{vmatrix} \lambda-1 & 2 & -2 \\ 2 & \lambda-1 & -2 \\ -2 & -2 & \lambda-1 \end{vmatrix}$

$= \underset{\uparrow \text{E.x}}{\cdots} = \lambda^3 - 3\lambda^2 - 9\lambda + 27$

$= (\lambda-3)^2(\lambda+3)$

Hence the eigenvalues of A:

$\underbrace{\lambda_1 = 3, \ \lambda_2 = 3,}_{\text{repeated because we have } "(\lambda-3)^2"} \ \lambda_3 = -3$

Step 2: Find the eigenvectors of A associated to $\lambda_1, \lambda_2, \lambda_3$.

① Find eigenvectors $\vec{v}_1, \vec{v}_2$ assoc. to λ_1, λ_2

Write $\vec{v}_1$ or $\vec{v}_2$ as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

we solve

$$(3I_3 - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 2x + 2y - 2z = 0 \\ 2x + 2y - 2z = 0 \\ -2x - 2y + 2z = 0 \end{cases}$$

$$\Rightarrow \quad 2x + 2y - 2z = 0$$

$$\Rightarrow \quad x + y - z = 0 \Rightarrow z = x + y$$

pick $x = 0$, $y = 1 \Rightarrow z = 1$.

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

pick $x = 1$, $y = 0 \Rightarrow z = 1$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Warning: There are many other choices of $\vec{v}_1, \vec{v}_2$. But we have to make $\vec{v}_1, \vec{v}_2$ linearly independent.

E.g.: Take

$x=1,\ y=0 \xRightarrow{z=x+y} z=1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$x=0,\ y=2 \Rightarrow z=2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

Okay!

$x=1,\ y=0 \Rightarrow z=1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$x=2,\ y=0 \Rightarrow z=2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

Not okay! NOT linearly independent

② Find eigenvector $\vec{v}_3$ assoc. to λ_3.

write $\vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

We solve

$$(\lambda_3 I_3 - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 2 & -2 \\ -2 & -4 & -2 \\ -2 & -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -4x+2y-2z=0 & ① \\ -2x-4y-2z=0 & ② \\ -2x-2y-4z=0 & ③ \end{cases}$$

Note ① + ② ⇒ ③. Hence only need to solve ①, ②

$$\begin{cases} -4x+2y-2z=0 \\ -2x-4y-2z=0 \end{cases}$$

$$\Rightarrow \begin{cases} -2x+y-z=0 \\ -x-2y-z=0 \end{cases}$$

pick $x=1 \Rightarrow y=1,\ z=-1$

$$\Rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Summarize:

$$\lambda_1 = 3 \longrightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3 \longrightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -3 \longrightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Step 3. Write down the general soln to the linear system:

$$\vec{x} = c_1 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

A math joke:

prove "$\frac{1}{\infty} = 0$" $\Longrightarrow$ "$\frac{1}{0} = \infty$"

Pf: We start with "$\frac{1}{\infty} = 0$"

Step 1: Rotate $\circlearrowleft$ by $90°$

$\frac{1}{\infty}$

$$-18 = 0$$

Step 2: Add 8

$$-10 = 8$$

Step 3: Rotate $\circlearrowright$ by $90°$

$$\frac{1}{0} = \infty$$

Remark: If you haven't done so, we strongly encourage you to do the CAPE evaluation for this course. Thanks!